

Spatiotemporal coupled-mode theory in dispersive media under a dynamic modulation

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Abstract

A simple and general formalism for mode coupling by a spatial, temporal or spatiotemporal perturbation in dispersive materials is developed. This formalism can be used for studying various linear and non-linear optical interactions involving a dynamic modulation of the interaction parameters such as non-reciprocal phenomena, time reversal of signals and spatiotemporal quasi phase matching.

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1. Introduction

Mode coupling is ubiquitous in physics, where different modes belonging to the same system or to different systems, can exchange energy through a coupling perturbation. There are many different formalisms of coupled-mode-theory in optics, describing a plethora of cases [1, 2, 3, 4, 5, 6]. The overwhelming majority of these formalisms relates to a spatial perturbation³. In recent years several works involving temporal or spatiotemporal coupling

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³We exclude cases such as temporal perturbation of a light field to couple energy eigenmodes of an atom or a molecule [7], as the modes themselves in this case are not propagating optical modes.

between optical modes were published. In this case, instead of the usual spatial perturbation (e.g. grating), a dynamic modulation is induced on one or more of the interaction parameters. It is important to distinguish between such a dynamic modulation to the dynamics already present by virtue of the oscillating optical fields. An analogue can be made with the regular spatial case, where the spatial modulation (such as a diffraction grating or a Bragg mirror) is distinguished from the spatial modulation embodied by the wavelength of the optical fields. Works which used such a dynamic modulation were concerned with nonreciprocal propagation [8, 9, 10], with time reversal of optical signals [11, 12, 13] and with phase matching of optical frequency conversion processes [14]. In those works coupled mode equations were used implicitly or explicitly, while dispersion or phase mismatch were not always taken into account. In 1990 Yariv has given the result for the specific case of two modes coupled by a travelling wave induced by the electro-optic effect [1, 15, 16] while recently, a formalism for spatiotemporal mode coupling in a system of parallel waveguides with no dispersion was derived [17]. These cases imply that to date there is no general formalism treating (dynamic modulation) for spatiotemporal mode coupling in general dispersive media. Here we develop such a formalism, which is general in the dispersive properties of the medium, number of modes and the geometry of the periodic spatiotemporal perturbation. This formalism can be applied to diverse cases, such as coupling between guided modes, coupling between free propagating and guided modes, and under non-depletion approximation also to nonlinear wave mixing and high harmonic generation.

2. Formalism

2.1. General spatiotemporal coupled mode analysis

We start with the wave equation for the electric field $\vec{E}(\vec{r}, t)$ in a non-magnetic medium [5]:

$$\nabla^2 \vec{E}(\vec{r}, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \vec{E}(\vec{r}, t) = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P}(\vec{r}, t) \quad (1)$$

where $c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the speed of light with ϵ_0 and μ_0 being the vacuum permittivity and vacuum permeability respectively and $\vec{P}(\vec{r}, t)$ is the material polarization vector.

The polarization model we analyse assumes a complete separation between a

linear transverse dispersive term $\chi^{(1)}(\vec{r}_\perp, t)$ and a non-dispersive spatiotemporal perturbation term $\Delta\varepsilon(\vec{r}, t)$. Here $\vec{r}_\perp = (x, y)$ and $\vec{r} = (x, y, z)$. For simplicity we also assume that both these terms are scalar. *Unlike in regular spatial couple mode theory in which $\Delta\varepsilon = \Delta\varepsilon(\vec{r})$, the time dependence of $\Delta\varepsilon(\vec{r}, t)$ in the present model allows to include a dynamic modulation as a coupling agent between modes.* Under these assumptions the polarization vector assumes the form [18]:

$$\begin{aligned}\vec{P}(\vec{r}, t) &= \vec{P}_L(\vec{r}_\perp, t) + \Delta\vec{P}(\vec{r}, t) = \\ &[\varepsilon_0\chi^{(1)}(\vec{r}_\perp, t) * + \Delta\varepsilon(\vec{r}, t) \cdot] \vec{E}(\vec{r}, t)\end{aligned}\quad (2)$$

where $*$ stands for convolution. Substituting the polarization term given by Eq. (2) in Eq. (1) and applying a Fourier transform in the time domain, results in:

$$\begin{aligned}\nabla^2 \hat{\vec{E}}(\vec{r}, \omega) + k^2(\vec{r}_\perp, \omega) \hat{\vec{E}}(\vec{r}, \omega) = \\ -\mu_0\omega^2[\Delta\hat{\varepsilon}(\vec{r}, \omega) * \hat{\vec{E}}(\vec{r}, \omega)]\end{aligned}\quad (3)$$

where $\hat{\vec{E}}(\vec{r}, \omega)$ stands for the Fourier transform of $\vec{E}(\vec{r}, t)$ and:

$$k^2(\vec{r}_\perp, \omega) = [1 + \hat{\chi}^{(1)}(\vec{r}_\perp, \omega)] \frac{\omega^2}{c_0^2} \quad (4)$$

In analogy with the case of having standard spatial perturbation [6], for the solution of the wave equation in the case of having spatiotemporal perturbation we invoke a spatiotemporal variation of constants in which the modes amplitudes are both spatially and temporally dependent:

$$\vec{E}(\vec{r}, t) = \sum_m a_m(z, t) \vec{E}_m(\vec{r}_\perp) e^{-i\beta_m z + i\omega_m t} \quad (5)$$

Here $\vec{E}_m(\vec{r}_\perp) e^{-i\beta_m z + i\omega_m t}$ are the normal modes of the system satisfying the unperturbed Helmholtz wave equation given by [6]:

$$[\nabla_\perp^2 + k^2(\vec{r}_\perp, \omega_m) - \beta_m^2] \vec{E}_m(\vec{r}_\perp) = 0 \quad (6)$$

In the frequency domain Eq. (5) can be written as:

$$\hat{\vec{E}}(\vec{r}, \omega) = \sum_m \hat{a}(z, \omega - \omega_m) \vec{E}_m(\vec{r}_\perp) e^{-i\beta_m z} \quad (7)$$

Assuming the coupling dynamic perturbation is periodic in both space and time, we can represent it with a Fourier series expansion:

$$\Delta\varepsilon(\vec{r}_\perp, t) = \sum_{p,q} \varepsilon_{p,q}(\vec{r}_\perp) e^{iq\Omega t - ipKz} \quad p, q \in \mathbb{Z} \quad (8)$$

Where Ω and K are the temporal and spatial fundamental frequencies of the modulation and the Fourier coefficients are given by:

$$\varepsilon_{p,q}(\vec{r}_\perp) = \frac{K\Omega}{4\pi^2} \int_{-\pi/K}^{\pi/K} dz \int_{-\pi/\Omega}^{\pi/\Omega} dt \Delta\varepsilon(\vec{r}_\perp, z, t) e^{iq\Omega t - ipKz} \quad (9)$$

In the frequency domain the perturbation is written as:

$$\Delta\hat{\varepsilon}(\vec{r}_\perp, \omega) = \sum_{p,q} \hat{\varepsilon}_{p,q}(\vec{r}_\perp) e^{-ipKz} \delta(\omega - q\Omega) \quad p, q \in \mathbb{Z} \quad (10)$$

Let us examine the expression:

$$k^2(\vec{r}_\perp, \omega) \hat{\vec{E}}(\vec{r}, \omega) \quad (11)$$

which appears in Eq. (3). Using Eq. (7) and assuming that the envelopes $\hat{a}(z, \omega - \omega_m)$ are narrow-band relative to their respective carrier frequencies ω_m such that they are spectrally isolated from each other we can expand Eq. (11) in the dispersion terms as:

$$\sum_m \left[k^2(\vec{r}_\perp, \omega_m) + \hat{D}(\omega) \right] \hat{a}(z, \omega - \omega_m) \vec{E}_m(\vec{r}_\perp) e^{-i\beta_m z} \quad (12)$$

where $\hat{D}(\omega) = \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\partial^l k^2(\vec{r}_\perp, \omega)}{\partial \omega^l} |_{\omega=\omega_m} (\omega - \omega_m)^l$ is the dispersion operator in the frequency domain.

We further make a functional approximation in which the dispersion terms are spatially independent. This way the spatial dependence of $k^2(\vec{r}_\perp, \omega)$ accounts for the guiding properties of the system, while the dispersive terms are defined within a volume of interest (such as the core of a wave-guide). Thus, $\frac{\partial^l}{\partial \omega^l} k^2(\vec{r}_\perp, \omega)|_{\omega=\omega_m} \cong \frac{\partial^l}{\partial \omega^l} k^2(\omega)|_{\omega=\omega_m}$. Substituting Eqs. (6-7), (10) and Eq. (11) into Eq. (3), while using a spatially slowly varying envelope approximation (SVEA) $\left| \frac{\partial^2}{\partial z^2} a_m(z, t) \right| \ll \left| \beta_m \frac{\partial}{\partial z} a_m(z, t) \right|$, we get:

$$\begin{aligned} & \sum_m \vec{E}_m(\vec{r}_\perp) e^{-i\beta_m z} \left[-2i\beta_m \frac{\partial}{\partial z} \hat{a}_m(z, \omega - \omega_m) + \hat{D}(\omega) \hat{a}_m(z, \omega - \omega_m) \right] \\ &= -\mu_0 \omega^2 \sum_{p,q} \hat{\epsilon}_{p,q}(\vec{r}_\perp) e^{-ipKz} \delta(\omega - \omega_n) * \\ & \sum_n \vec{E}_n(\vec{r}_\perp) \hat{a}_n(z, \omega - \omega_n) e^{-i\beta_n z} \end{aligned} \quad (13)$$

We use the usual normalization for a 1 [W] power flow in the propagation direction which results in the orthogonality relation [6]:

$$\begin{aligned} & \langle \vec{E}_m(\vec{r}_\perp) | \vec{E}_n(\vec{r}_\perp) \rangle \\ &= \int d\vec{r}_\perp \vec{E}_m(\vec{r}_\perp)^* \vec{E}_n(\vec{r}_\perp) = \frac{2\omega_m \mu}{|\beta_m|} \delta_{mn} \end{aligned} \quad (14)$$

where $\langle \cdot | \cdot \rangle$ stands for inner product and δ_{mn} is either the Kronecker delta function for bound modes or the Dirac delta function for radiation modes. With this orthogonality condition we project Eq. (13) on a specific mode $\langle m |$ to get:

$$\begin{aligned} & \frac{2\omega_m}{|\beta_m|} e^{-i\beta_m z} \left[-2i\beta_m \frac{\partial}{\partial z} \hat{a}_m(z, \omega - \omega_m) + \hat{D}(\omega) \hat{a}_m(z, \omega - \omega_m) \right] = \\ & -\omega^2 \sum_n \sum_{p,q} C_{mn}^{pq} e^{-i(\beta_n + pK)z} \hat{a}_n(z, \omega - \omega_n) * \delta(\omega - q\Omega) \end{aligned} \quad (15)$$

with the coupling coefficient:

$$C_{mn}^{pq} = \int d\vec{r}_\perp \vec{E}_m(\vec{r}_\perp)^* \hat{\epsilon}_{p,q}(\vec{r}_\perp) \vec{E}_n(\vec{r}_\perp) \quad (16)$$

Transforming back to the time domain Eq. (15) takes the form:

$$\begin{aligned} & \left[\frac{\partial}{\partial z} - \frac{1}{2i\beta_m} D(t) \right] a_m(z, t) \\ &= \frac{|\beta_m|}{4i\omega_m\beta_m} \sum_n \sum_{p,q} C_{mn}^{pq} e^{-i(\beta_n - \beta_m + pK)z} e^{-i\omega_m t} \frac{\partial^2}{\partial t^2} a_n(z, t) e^{i(\omega_n + q\Omega)t} \end{aligned} \quad (17)$$

where $D(t) = \sum_{l=1}^{\infty} \frac{(-i)^l}{l!} \frac{\partial^l k^2(\omega)}{\partial \omega^l} \Big|_{\omega=\omega_m} \frac{\partial^l}{\partial t^l} = \sum_{l=1}^{\infty} \frac{2k(\omega_m)(-i)^l}{l!} \left(\frac{\partial^l k(\omega)}{\partial \omega^l} \Big|_{\omega=\omega_m} \right) \frac{\partial^l}{\partial t^l}$ is the dispersion operator in the time domain [19].

Eq. (17) is the major result of this paper. Unlike in regular spatial perturbation mode coupling, this result includes the effect of a periodic applied dynamic modulation through the appearance of the $q\Omega$ temporal frequencies.

If we now further assume a second-order temporal SVEA: $\left| \frac{\partial^2}{\partial t^2} a_n(z, t) \right| \ll \left| \omega_n \frac{\partial}{\partial t} a_n(z, t) \right| \ll \left| \omega_n^2 a_n(z, t) \right|$ and that $|q\Omega|$ is much smaller than ω_n we are left with:

$$\begin{aligned} & \left[\frac{\partial}{\partial z} - \frac{1}{2i\beta_m} D(t) \right] a_m(z, t) \\ &= \kappa_m \sum_n \sum_{p,q} C_{mn}^{pq} a_n(z, t) e^{i\Delta\omega_{m,n,q}t} e^{-i\Delta k_{m,n,q}z} \end{aligned} \quad (18)$$

with $\kappa_m = \frac{i|\beta_m|(\omega_n + q\Omega)^2}{4\omega_m\beta_m}$ and where $\Delta k_{m,n,q} = \beta_n - \beta_m + pK$ is the momentum phase mismatch and $\Delta\omega_{m,n,q} = \omega_n - \omega_m + q\Omega$ is the energy phase mismatch. This form is appealing as the spatial and temporal phase mismatch terms appear with the same functional form. The appearance of the temporal phase mismatch is the result of allowing for a dynamic modulation. This term does not appear in regular - spatial coupled mode theory.

2.2. The special case of two coupled modes under first order dispersion approximation

Eq. (18) describes a general case of mode coupling caused by a temporal and/or spatial perturbations in a dispersive media. In many cases of interest there are only two coupled modes (there might be other modes but the coupling with these other modes is negligible) and only the first dispersion term is important. Let the two coupled mode be designated with the indices

1 and 2. We assume that the coupling is made through a specific (p, q) order of the perturbation. In this case the coupled mode equations reduce to:

$$\begin{aligned} & \left(\frac{\partial}{\partial z} + \frac{k(\omega_1)}{\beta_1} \frac{1}{v_g(\omega_1)} \frac{\partial}{\partial t} \right) a_1(z, t) \\ &= \kappa_1 C_{12}^{p,q} a_2(z, t) e^{i\Delta\omega t} e^{-i\Delta k z} \end{aligned} \quad (19)$$

$$\begin{aligned} & \left(\frac{\partial}{\partial z} + \frac{k(\omega_2)}{\beta_2} \frac{1}{v_g(\omega_2)} \frac{\partial}{\partial t} \right) a_2(z, t) \\ &= \kappa_2 (C_{12}^{p,q})^* a_1(z, t) e^{-i\Delta\omega t} e^{i\Delta k z} \end{aligned} \quad (20)$$

with the phase mismatch components $\Delta k = \beta_2 - \beta_1 + pK$ and $\Delta\omega = \omega_2 - \omega_1 + q\Omega$. We have used the relation $C_{21}^{-p,-q} = (C_{12}^{p,q})^*$ which is easily derived from Eq. (16). In addition we used $\frac{\partial}{\partial\omega} k^2(\omega)|_{\omega=\omega_m} = 2\frac{k(\omega_m)}{v_g(\omega_m)}$ with the usual definition of the group velocity $v_g^{-1}(\omega) = \frac{\partial}{\partial\omega} k(\omega)$.

We would like to comment that these last two equations can be further reduced to known cases in the literature. With no temporal modulation: $\Delta\omega = 0$ and $q\Omega = 0$, they reduce to regular spatial coupled mode equations [1, 4]. For the case of nonlinear frequency conversion under the non-depletion approximation: With temporal modulation: Eq. (20) is virtually identical in form to Eq. (2) in Ref.[14]; with no temporal modulation this same equation is reduced to the form found in Eq. (2.7.11) in Ref.[21].

3. Discussion

In this work we developed a general coupled-mode formalism under the presence of a dynamic modulation leading to a temporal or a spatiotemporal perturbative coupling of modes. *This dynamic modulation is absent from regular spatial optical coupled mode theory.* This generalization can not only describe known phenomena such as optical non-reciprocity and time reversal but can also leads to new ones - such as the formation of a breathing soliton pair with no nonlinearity [20].

Our model is not restricted to any specific system or mechanism for inducing the perturbation. It applies to any number of modes, as well as to a continuum of modes. It includes high order dispersion terms. The model requires that the coupling perturbation can be approximated as non-dispersive,

compared to the linear susceptibility term, as is explicit in Eq.(2). This condition is satisfied for example when the perturbation is applied through the electrooptic effect. This can be proved using Miller's rule [21] applied to the electrooptic nonlinear polarization term. Another implicit required condition is that the spatial perturbation and temporal modulations are much slower than those of the optical frequencies of the relevant modes. Otherwise the problem becomes one of effective medium approximation.

4. References

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